1011 – I2 - 1

Ako su

$\Phi = \mathrm{r}\sin(\pi\varphi + \vartheta)$; $\vec{F} = r^2 \sin\vartheta\,\hat{r} + \mathrm{r}\cos\varphi\,\hat{\vartheta} + \cos\vartheta\,\hat{\varphi}$; $M = \log_2 \sin \mathrm{x}^{\mathrm{y+z}}$; $\vec{E} = \frac{\rho}{z}\hat{\rho}$

gdje su (r,ϑ,φ) Kartezijeve koordinate, (r,ϑ,φ) sferne koordinate, a (ρ,φ,z) cilindrične, odredite u pripadnom koordinatnom sustavu

(a) $\nabla\Phi$; (b) $\nabla\,\vec{F}$; (c) $\nabla\times\,\vec{E}$; (d) $\mathrm{rot}(\nabla\mathrm{M})$.

U ortogonalnim koordinatnim sustavima dani su opći izrazi za gradijent, divergenciju, rotaciju i laplasijan

$$\nabla\Phi = \sum_{i=1}^{3} \frac{1}{h_i}\cdot\frac{\partial\Phi}{\partial q_i}\widehat{q_i} \qquad \nabla\vec{F} = \frac{1}{h_1h_2h_3}\sum_{i(j,k)}\frac{\partial}{\partial q_i}\left(h_jh_kF_i\right)$$

$$\nabla\times\vec{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\widehat{q_1} & h_2\widehat{q_2} & h_3\widehat{q_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \qquad \Delta = \frac{1}{h_1h_2h_3}\sum_{i(j,k)}\frac{\partial}{\partial q_i}\left(\frac{h_jh_k}{h_i}\cdot\frac{\partial}{\partial q_i}\right)$$

gdje su h_i Lameovi koeficijenti koji iznose za

Kartezijev koordinatni sustav	cilindrični koordinatni sustav	sferni koordinatni sustav
$h_1 = 1$	$h_1 = 1$	$h_1 = 1$
$h_2 = 1$	$h_2 = r$	$h_2 = \rho$
$h_3 = 1$	$h_3 = r\sin\vartheta$	$h_3 = 1$

Gradijent u sfernom

$$\nabla\Phi = \frac{1}{h_1}\cdot\frac{\partial\Phi}{\partial q_1}\widehat{q_1} + \frac{1}{h_2}\cdot\frac{\partial\Phi}{\partial q_2}\widehat{q_2} + \frac{1}{h_3}\cdot\frac{\partial\Phi}{\partial q_3}\widehat{q_3}$$

$$\nabla\Phi = \frac{1}{1}\cdot\frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\cdot\frac{\partial\Phi}{\partial\vartheta}\hat{\vartheta} + \frac{1}{r\sin\vartheta}\cdot\frac{\partial\Phi}{\partial\varphi}\hat{\varphi}$$

$$\nabla\Phi = \frac{1}{1}\cdot\sin(\pi\varphi+\vartheta)\,\hat{r} + \frac{1}{r}\cdot r\cos(\pi\varphi+\vartheta)\,\hat{\vartheta} + \frac{1}{r\sin\vartheta}\cdot r\pi\cos(\pi\varphi+\vartheta)\,\hat{\varphi}$$

$$\nabla\Phi = \sin(\pi\varphi+\vartheta)\,\hat{r} + \cos(\pi\varphi+\vartheta)\,\hat{\vartheta} + \frac{\pi}{\sin\vartheta}\cos(\pi\varphi+\vartheta)\,\hat{\varphi}$$

Divergencija u sfernom

$$\nabla\vec{F} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q_1}(h_2h_3\cdot F_1) + \frac{\partial}{\partial q_2}(h_3h_1\cdot F_2) + \frac{\partial}{\partial q_3}(h_1h_2\cdot F_3)\right\}$$

$$\nabla\vec{F} = \frac{1}{r^2\sin\vartheta}\left\{\frac{\partial}{\partial r}(r^2\sin\vartheta\cdot r^2\sin\vartheta) + \frac{\partial}{\partial\vartheta}(r\sin\vartheta\cdot r\cos\varphi) + \frac{\partial}{\partial\varphi}(r\cdot\cos\vartheta)\right\}$$

$$\nabla\vec{F} = \frac{1}{r^2\sin\vartheta}(4r^3\sin^2\vartheta + r^2\cos\vartheta\cdot\cos\varphi + 0)$$

$$\nabla\vec{F} = 4r\sin\vartheta + \mathrm{ctg}\,\vartheta\cdot\cos\varphi$$

Rotacija u cilindričnom

$$\nabla \times \vec{E} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \widehat{q_1} & h_2 \widehat{q_2} & h_3 \widehat{q_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

$$\nabla \times \vec{E} = \frac{1}{\rho} \begin{vmatrix} 1 \cdot \hat{\rho} & \rho \cdot \hat{\varphi} & 1 \cdot \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 1 \cdot \frac{\rho}{z} & \rho \cdot 0 & 1 \cdot 0 \end{vmatrix} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ \frac{\rho}{z} & 0 & 0 \end{vmatrix}$$

$$\nabla \times \vec{E} = \frac{\hat{\rho}}{\rho} \left[\frac{\partial}{\partial \varphi}(0) - \frac{\partial}{\partial z}(0) \right] + \hat{\varphi} \left[\frac{\partial}{\partial z}\left(\frac{\rho}{z}\right) - \frac{\partial}{\partial \rho}(0) \right] + \frac{\hat{z}}{\rho} \left[\frac{\partial}{\partial \rho}(0) - \frac{\partial}{\partial \varphi}\left(\frac{\rho}{z}\right) \right]$$

$$\nabla \times \vec{E} = \frac{\hat{\rho}}{\rho}[0 - 0] + \hat{\varphi} \left[-\frac{\rho}{z^2} - 0 \right] + \frac{\hat{z}}{\rho}[0 - 0]$$

$$\nabla \times \vec{E} = -\frac{\rho}{z^2} \hat{\varphi}$$

Identitet $\nabla \times (\nabla \Phi) = 0$

$$\mathrm{rot}(\nabla \mathrm{M}) = 0$$

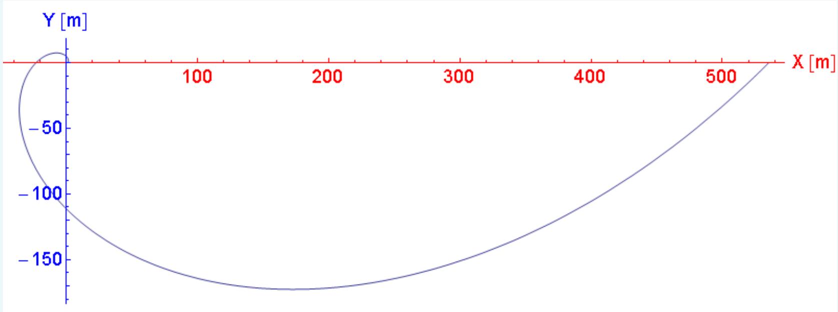
1011 – I2 - 2

Odredite rad sile (koordinate su izražene u metrima)

$$\vec{F} = \frac{x\hat{\imath} + y\hat{\jmath}}{(x^2 + y^2)^{\frac{3}{2}}} [\mathrm{Nm}^2]$$

na putu od točke (1,0)m do točke ($e^{2\pi}$,0)m po krivulji

$$\vec{r}(t) = (e^t \cos t)\hat{\imath} + (e^t \sin t)\hat{\jmath}$$



Rad sile duž krivulje C iznosi

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot (dx\hat{\imath} + dy\hat{\jmath})$$

Parametrizacija

Kako je putanja

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

već zadana parametrizirana

$$\vec{r}(t) = (e^t \cos t)\hat{\imath} + (e^t \sin t)\hat{\jmath}$$

zaključujemo

$$x = e^t \cos t \Longrightarrow dx = (e^t \cos t - e^t \sin t)dt$$
$$y = e^t \sin t \Longrightarrow dy = (e^t \sin t + e^t \cos t)dt$$

Prema tome sila je

$$\vec{F} = \frac{e^t \cos t\, \hat{\imath} + e^t \sin t\, \hat{\jmath}}{((e^t \cos t)^2 + (e^t \sin t)^2)^{\frac{3}{2}}} = \frac{e^t \cos t\, \hat{\imath} + e^t \sin t\, \hat{\jmath}}{e^{3t}} \;[\mathrm{N}]$$

Granice integracije

Kako ćemo integrirati po t, potrebno je odrediti i njegove granične vrijednosti:

Početna točka A(1,0)

$$\begin{matrix}(1) & x = e^t \cos t = 1 \\ (2) & y = e^t \sin t = 0\end{matrix} \xRightarrow{(2)/(1)} \mathrm{tg}\, t = 0 \Longrightarrow \begin{matrix} t = k\pi \\ k \in \mathbb{Z}\end{matrix}$$

$$\begin{matrix}(1) & \Longrightarrow t = \ln(1/\cos t) \\ (\mathrm{RJ}) & \Longrightarrow \cos(k\pi) = \pm 1\end{matrix} \xRightarrow{Arg\,(ln)>0} t = \ln\left(\frac{1}{1}\right) = 0$$

$$t_A = 0$$

Krajnja točka B($e^{2\pi}$,0)

$$\begin{matrix}(1) & x = e^t \cos t = e^{2\pi} \\ (2) & y = e^t \sin t = 0\end{matrix} \xRightarrow{(2)/(1)} \mathrm{tg}\, t = 0 \Longrightarrow \begin{matrix} t = k\pi \\ k \in \mathbb{Z}\end{matrix}$$

$$\begin{matrix}(1) & \Longrightarrow t = \ln(e^{2\pi}/\cos t) \\ (\mathrm{RJ}) & \Longrightarrow \cos(k\pi) = \pm 1\end{matrix} \xRightarrow{Arg\,(ln)>0} t = \ln(e^{2\pi})$$

$$t_B = 2\pi$$

Prema tome rad iznosi

$$W = \int_C \vec{\mathrm{F}} \cdot (\mathrm{dx}\hat{\imath} + \mathrm{dy}\hat{\jmath}) = \int_{t_A}^{t_B} \left[\frac{e^t \cos t\, \hat{\imath} + e^t \sin t\, \hat{\jmath}}{e^{3t}}\right] \cdot [(e^t \cos t - e^t \sin t)dt\hat{\imath} + (e^t \sin t + e^t \cos t)dt\hat{\jmath}]$$

$$W = \int_0^{2\pi} \frac{e^t \cos t\,(e^t \cos t - e^t \sin t) + e^t \sin t\,(e^t \sin t + e^t \cos t)}{e^{3t}}\,\mathrm{dt}$$

$$W = \int_0^{2\pi} \frac{e^{2t} \cos^2 t - e^{2t} \sin t \cos t + e^{2t} \sin^2 t + e^{2t} \sin t \cos t}{e^{3t}}\,\mathrm{dt}$$

$$W = \int_0^{2\pi} \frac{e^{2t}(\cos^2 t + \sin^2 t)}{e^{3t}}\,\mathrm{dt}$$

$$W = \int_0^{2\pi} e^{-t}\mathrm{dt} = -\int_0^{2\pi} e^{-t}\mathrm{d(-t)}$$

$$W = -e^{-t}\Big|_{\varphi=0}^{\varphi=2\pi} = -e^{-2\pi} + e^{-0}$$

$$\mathrm{W} = (1 - e^{-2\pi})$$

$$\mathrm{W} = \frac{e^{2\pi} - 1}{e^{2\pi}}\ \mathrm{J}$$

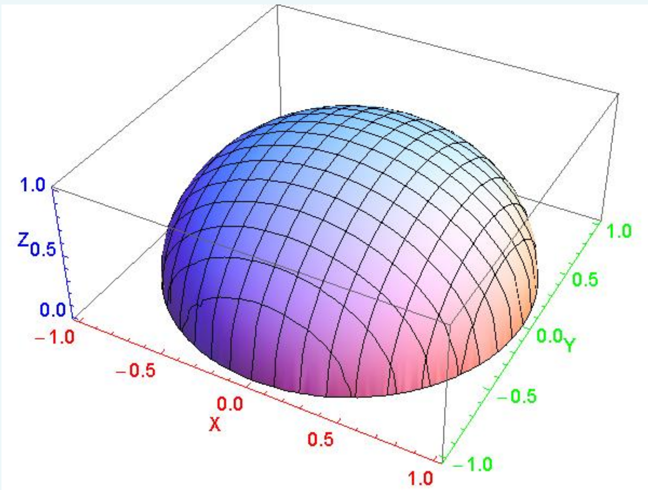
1011 - I2 - 3

Odredite masu tijela u obliku polovice kugle (koordinate su izražene u metrima)

$$x^2 + y^2 + z^2 \leq 1\, m^2$$

iznad xy ravnine, ako ima promjenjivu gustoću

$$\rho = 7700\, kgm^{-3} + 160z\, kgm^{-4}.$$



Masa tijela iznosi

$$M = \int_V \rho dV$$

Granice integracije

Integriramo po kuglinoj $(x^2 + y^2 + z^2 \leq r^2)$ polovici iznad xy ravnine $(\mathrm{z} \geq 0)$pa je najlakše integraciju vršiti u sfernom koordinatnom sustavu.

koordinate:	element volumena:	zadana kuglina polovica:
$x = r \sin\vartheta \cos\varphi$	$dV = r^2 \sin\vartheta\, dr d\vartheta d\varphi$	$0 \leq r \leq 1$
$y = r \sin\vartheta \sin\varphi$		$0 \leq \vartheta \leq \pi/2$
$z = r \cos\vartheta$		$0 \leq \varphi \leq 2\pi$

$$M = \int_V \rho dV = \iiint_V \rho r^2 \sin\vartheta\, dr d\vartheta d\varphi = \iiint_V (7700 + 160 r \cos\vartheta) r^2 \sin\vartheta\, dr d\vartheta d\varphi = M_1 + M_2$$

$$M = 7700 \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{2\pi} d\varphi \int_0^1 r^2 \sin\vartheta\, dr + 160 \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{2\pi} d\varphi \int_0^1 r^3 \sin\vartheta \cos\vartheta\, dr = M_1 + M_2$$

$$M_1 = 7700 \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{2\pi} d\varphi \int_0^1 r^2 \sin\vartheta\, dr = 7700 \int_0^{\frac{\pi}{2}} \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi \left[\frac{\mathrm{r}^3}{3}\right]\Bigg|_{\mathrm{r}=0}^{\mathrm{r}=1} = \frac{7700}{3} \int_0^{\frac{\pi}{2}} \sin\vartheta\, d\vartheta [\varphi]\big|_{\varphi=0}^{\varphi=2\pi}$$

$$M_1 = \frac{15400}{3}\pi \int_0^{\frac{\pi}{2}} \sin\vartheta\, d\vartheta = \frac{15400}{3}\pi[-\cos\vartheta]\big|_{\vartheta=0}^{\vartheta=\frac{\pi}{2}} = \frac{15400}{3}\pi[-0+1] = \frac{15400}{3}\pi \text{ kg}$$

$$M_2 = 160\int_0^{\frac{\pi}{2}} d\vartheta \int_0^{2\pi} d\varphi \int_0^1 r^3 \sin\vartheta \cos\vartheta \, dr = 160\int_0^{\frac{\pi}{2}} \frac{\sin(2\vartheta)}{2} \, d\vartheta \cdot [\varphi]\Big|_{\varphi=0}^{\varphi=2\pi} \cdot \left[\frac{\mathrm{r}^4}{4}\right]\Bigg|_{\mathrm{r}=0}^{\mathrm{r}=1}$$

$$M_2 = 80\int_0^{\frac{\pi}{2}} \sin(2\vartheta) \, d\vartheta \cdot [2\pi] \cdot \left[\frac{1}{4}\right] = \frac{80\pi}{2}\int_0^{\frac{\pi}{2}} \sin(2\vartheta) \cdot \frac{d(2\vartheta)}{2} = \frac{80\pi}{2}\int_0^{\frac{\pi}{2}} \sin(2\vartheta) \cdot \frac{d(2\vartheta)}{2}$$

$$M_2 = -20\pi \cdot [\cos(2\vartheta)]\Big|_{\vartheta=0}^{\vartheta=\frac{\pi}{2}} = -20\pi \cdot [-1-1] = 40\pi \text{ kg}$$

$$M = M_1 + M_2 = \frac{15400}{3}\pi \text{ kg} + 40\pi \text{ kg} = \frac{15\,520}{3}\pi \text{ kg}$$

$$M = \frac{15\,520}{3}\pi \text{ kg}$$

1011 - I2 – 4

Od svih mogućih kvadara, definiranih nejednadžbama $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq 1$, odredite onaj za koji je ukupni tok polja

$$\vec{F} = (-x^2 - 4xy)\hat{\imath} - 6yz\hat{\jmath} + 12z\hat{k}$$

prema vani kroz svih 6 površina najveći. Koliko iznosi najveći tok?

Volumen kvadra V zatvoreno je područje u prostoru čiji je rub orjentabilna po dijelovima glatka ploha $\vec{S}$, koja ne presjeca samu sebe, a $\vec{F} = (-x^2 - 4xy)\hat{\imath} - 6yz\hat{\jmath} + 12z\hat{k}$ vektorsko polje klase $C^1(V)$ pa možemo primijeniti **Gaussov teorem o divergenciji**

$$\oiint_S \vec{\mathrm{F}} \cdot d\vec{S} = \iiint_V \nabla\vec{\mathrm{F}} \cdot dV$$

Divergencija polja $\nabla\vec{\mathrm{F}}$

$$\nabla\vec{F} = \frac{\partial \mathrm{F_x}}{\partial \mathrm{x}} + \frac{\partial \mathrm{F_y}}{\partial \mathrm{y}} + \frac{\partial \mathrm{F_z}}{\partial \mathrm{z}} = -2\mathrm{x} - 4\mathrm{y} - 6\mathrm{z} + 12$$

Granice integracije

Kako su granične plohe ravnine, najjednostavnije je računati s Kartezijevim koordinatama (x, y, z). Granice integracije zadane su definicijom kvadara

$$0 \leq x \leq a$$
$$0 \leq y \leq b$$
$$0 \leq z \leq 1$$

Sada imamo sve potrebne elemente za integriranje pa možemo izračunati ukupni tok polja

$$\oiint_S \vec{\mathrm{F}} \cdot d\vec{S} = \iiint_V \nabla\vec{\mathrm{F}} \cdot dV = \int_0^a dx \int_0^b dy \int_0^1 (-2x - 4y - 6z + 12) \cdot dz =$$

$$= \int_0^a dx \int_0^b dy(-2xz - 4yz - 3z^2 + 12z)\Big|_{z=0}^{z=1} = \int_0^a dx \int_0^b dy(-2x - 4y - 3 + 12) =$$

$$= \int_0^a dx(-2xy - 2y^2 + 9y)\Big|_{y=0}^{y=b} = \int_0^a dx(-2xb - 2b^2 + 9b) =$$

$$= (-x^2b - 2b^2x + 9bx)\Big|_{x=0}^{x=a} = -a^2b - 2ab^2 + 9ab = f(a, b)$$

Dakle, traženi je tok funkcija dviju varijabli (duljina stranica kvadra a, b) pa ćemo kvadar za koji je tok maksimalan odrediti tražeći maksimum te funkcije.

NUŽNI UVJETI

Ekstreme funkcije $f(x_1, \dots, x_n)$ ispitujemo u točkama $x_0 = (x_{01}, \dots, x_{0n})$ u kojima nije definirana df i onima koje dobijemo rješavajući sustav

$$\left(\frac{\partial f}{\partial x_i}\right)_{x_0} = 0 \qquad (\forall i = 1, \dots, n)$$

Nađemo sve parcijalne derivacije prvog reda i izjednačimo s nulom

$$\frac{\partial f}{\partial a} = -2ab - 2b^2 + 9b = 0 \qquad \Longrightarrow \qquad -b(2b + 2a - 9) = 0 \qquad (1)$$

$$\frac{\partial f}{\partial b} = -a^2 - 4ab + 9a = 0 \qquad \Longrightarrow \qquad -a(a + 4b - 9) = 0 \qquad (2)$$

Iz (2) i (1) imamo 4 mogućnosti iz kojih nalazimo 4 kandidata za točke ekstrema

$a = 0$	$a = 0$	$a + 4b - 9 = 0$	$a + 4b = 9$
$b = 0$	$2b + 2a - 9 = 0$	$\mathrm{b} = 0$	$2a + 2b = 9$
$a = 0$	$a = 0$	$a = 9$	$a = 3$
$b = 0$	$\mathrm{b} = 4.5$	$\mathrm{b} = 0$	$\mathrm{b} = 1.5$
$\boldsymbol{K_1(0,0)}$	$\boldsymbol{K_2(0,4.5)}$	$\boldsymbol{K_3(9,0)}$	$\boldsymbol{K_4(3,1.5)}$

DOVOLJNI UVJETI

Odredimo determinante za funkciju $f(x_1, x_2, \dots, x_n)$:

$$\Delta_r = \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{vmatrix} \quad ; \quad 1 \le r \le n \quad ; \quad a_{ij} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{x_0}$$

Funkcija f u točki x_0:

- ima MINIMUM, ako je
  $(\Delta_r > 0)\ (\forall r)$
- ima MAKSIMUM, ako je
  $\begin{cases} (\Delta_r > 0)(\forall r = 2k) \\ (\Delta_r < 0)\ (\forall r = 2k - 1) \end{cases}$
- NEMA EKSTREMA, ako je
  $(\Delta_r \neq 0)\ \wedge \overline{(1)} \wedge \overline{(2)}$
- NEMA ODLUKE, ako
  $(\exists \Delta_r = 0)$
  pa su potrebna dodatna ispitivanja (df)

Kako bismo provjerili, radi li se o ekstremima, moramo odrediti druge parcijalne derivacije

$$\frac{\partial^2 f}{\partial a^2} = -2b \qquad \frac{\partial^2 f}{\partial a \partial b} = -2a - 4b + 9$$

$$\frac{\partial^2 f}{\partial b \partial a} = -2a - 4\mathrm{b} + 9 \qquad \frac{\partial^2 f}{\partial b^2} = -4a$$

pa imamo

$$\Delta_1 = |-2b| = -2b \qquad \Delta_2 = \begin{vmatrix} -2b & -2a - 4b + 9 \\ -2a - 4b + 9 & -4a \end{vmatrix} = 8ab - (2a + 4b - 9)^2$$

koje u dobivenim točkama iznose

$\Delta_1(K_1) = 0$	$\Delta_1(K_2) = -9$	$\Delta_1(K_3) = 0$	$\Delta_1(K_4) = -3$
$\Delta_2(K_1) = -81$	$\Delta_2(K_2) = -81$	$\Delta_2(K_3) = -81$	$\Delta_2(K_4) = 27$
nema odluke	**nema ekstrema**	**nema odluke**	**maksimum**

Kako je $f(K_1) = f(K_3) = 0 < f(K_4) = 13.5$, a nama treba maksimalni tok, ne trebamo dodatno ispitivati.

Znači, za kvadar (a=3, b=1.5, c=1) imamo maksimalni tok koji iznosi

$$f(K_4) = 13.5$$

1011 - I2 - 5

Dokažite:

a) Tenzor ranga 2 može se napisati kao zbroj simetričnog i antisimetričnog tenzora.

b) Ako je rang tenzora $\tilde{A}$ i $\tilde{B}$ naznačen brojem indeksa u relacija $K^{ijkl} A_{ij} = B^{kl}$, koja vrijedi u svim (zarotiranim) Kartezijevim sustavima, tada je $\tilde{K}$ tenzor ranga 4.

a)

Kako su dokazi za kovarijantne, kontravarijantne i miješane tenzore slični, dokazat ćemo za kontravarijantni:

$$T^{ij} = \frac{1}{2}T^{ij} + \frac{1}{2}T^{ij}$$

Dodamo li i oduzmemo $T^{ji}/2$ nećemo promijeniti tenzor

$$T^{ij} = \frac{T^{ij} + T^{ji}}{2} + \frac{T^{ij} - T^{ji}}{2}$$

Zamjena indeksa odgovara transponiranju matrične reprezentacije vektora pa ako transponiramo prvi sumand

$$\left(\frac{T^{ij} + T^{ji}}{2}\right)^{\tau} = \frac{\left(T^{ij}\right)^{\tau} + \left(T^{ji}\right)^{\tau}}{2} = \frac{T^{ji} + T^{ij}}{2} = \frac{T^{ij} + T^{ji}}{2}$$

dobijemo isto, što znači

$$\frac{T^{ij} + T^{ji}}{2}$$

simetrični je tenor. Transponiranjem drugog dobijemo

$$\left(\frac{T^{ij} - T^{ji}}{2}\right)^{\tau} = \frac{\left(T^{ij}\right)^{\tau} - \left(T^{ji}\right)^{\tau}}{2} = \frac{T^{ji} - T^{ij}}{2} = -\frac{T^{ij} - T^{ji}}{2}$$

što znači

$$\frac{T^{ij} + T^{ji}}{2}$$

antisimetrični je tenzor pa tenzor ranga 2 možemo napisati kao zbroj simetričnog i antisimetričnog tenzora.

b)

Dana relacija vrijedi u vrijedi u svim (zarotiranim) Kartezijevim sustavima pa ćemo je napisati u crtkanome

$K'^{ijkl} A'_{ij} = B'^{kl} = a^k_m a^l_n B^{mn} =$ // B transformiramo u necrtkani sustav

$= a^k_m a^l_n \left(K^{opmn} A_{op}\right) =$ // Primijenimo kvocijentno pravilo

$= a^k_m a^l_n \left(K^{opmn} a^i_o a^j_p A'_{ij}\right) =$ // A transformiramo u crtkani

$= a^i_o a^j_p a^k_m a^l_n K^{opmn} A'_{ij}$

$K'^{ijkl} A'_{ij} = a^i_o a^j_p a^k_m a^l_n K^{opmn} A'_{ij}$ //Izjednačimo početak i kraj jednakosti

$\left(K'^{ijkl} - a^i_o a^j_p a^k_m a^l_n K^{opmn}\right) A'_{ij} = 0$ // A općenit pa je izraz u zagradi 0

$K'^{ijkl} = a^i_o a^j_p a^k_m a^l_n K^{opmn}$ // Način transformacije tenzora 4. ranga